Perihelion precession and deflection of light in gravitational field of wormholes

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Abstract

Quite exotic relativistic objects known as wormholes are hypothetical candidates for central machine of active galactic nuclei as well as black holes. We find the magnitude of the perihelion precession and the deflection of light in gravitational field of a wormhole and compare them with those for a black hole. The impact parameter is taken to be much larger than the wormhole throat size. We show that the relative difference between results for a black hole and a wormhole may be significant and amount to tens of percent.

1 Introduction

Einstein and Rosen [1] were first to propose nontrivial topological configurations known as bridges in the framework of the General Relativity (GR). Nowadays, nontrivial configurations known as wormholes are widely discussed. Moreover, the interest to these objects increased last decade.

There are several types of wormholes depending on their topology [2]. In this paper we concentrate on so-called traversable Lorentzian wormholes [3]. Bodies can freely pass through a traversable wormhole throat in both directions and there are no event horizons in these objects. It have been recently proposed [5] that observed active galactic nuclei (AGN) and some other high-energy objects in the Universe may be the former or present entrances to wormholes. In light of this suggestion it is interesting to calculate some effects which could possibly distinguish a wormhole from a black hole in an experiment. Some of the effects, such as magnetic field of a wormhole [5], transition of light through a wormhole throat [6], lensing by wormholes [7], accretion onto wormholes [8] etc. have been proposed.

In this paper we calculate two classical GR effects, the precession of perihelion (we still use "perihelion" for the closest orbit point) and deflection of light in the gravitational field of a wormhole with central symmetry, and compare them with those for a black hole. Deflection of light was also calculated in the paper [6], but with no relation to experiment. For our calculations we use an explicit metrics of a wormhole made of certain matter [5]. We also make a remark on the third GR effect, gravitational redshift.

The outline of the paper is as follows. In Sect. 2 we present the equations of motion in a spherically symmetric wormhole field. In Sect. 3 we find the perihelion precession magnitude in two different ways. In Sect. 4 we find the light deflection magnitude. And in Sect. 5 we discuss the results and draw the conclusions comparing the results with those for a black hole.

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2 Equations of motion

A quite general expression for the wormhole metrics with central symmetry is [3]:

$$ds^2 = e^{2\phi(r)}dt^2 - \frac{dr^2}{1 - \dfrac{b(r)}{r}} - r^2 d\Omega^2, \tag{1}$$

where $d\Omega^2 = \sin^2\theta d\varphi^2 + d\theta^2$, $b(r)$ and $\phi(r)$ are the functions of a radial coordinate only.

In order to construct a wormhole a phantom matter ($p < -\varepsilon$) with anisotropic equation of state is required [4]. We consider the following equation of state [5]:

$$1 + \delta = -p_{\|}/\varepsilon = p_{\perp}/\varepsilon, \tag{2}$$

where $\delta > 0$. The possibility of realization of this equation of state is not clear yet, nevertheless it is widely considered. Under these assumptions on what the wormhole is made of, in the paper [5] the metrics was found explicitly:

$$\begin{aligned} ds^2 &= \left(1 - \frac{r_h}{r}\right)^{2+2\delta} c^2 dt^2 - \\ &- \frac{dr^2}{1 - \dfrac{r_h}{r}\left[1 + \left(1 - \dfrac{r_h}{r}\right)^{1-\delta}\right]} - r^2 d\Omega^2. \end{aligned} \tag{3}$$

In the limit case of $\delta = 0$, the metrics turns into the Reissner-Nordström (RN) metrics, and $r_h = GM/c^2$ is a gravitational radius of a corresponding RN black hole.

We use the metrics (3) to derive the equations of motion and then to carry out calculations of the perihelion precession and the deflection of light. The metrics (3) possesses spherical symmetry, hence, we can consider only the equatorial plane motion and set $\theta = \pi/2$. Thus, $d\Omega^2 = d\varphi^2$.

We can write the Hamilton-Jacobi equation for a particle with the mass m as following:

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} = m^2c^2 \qquad\qquad (i, k = 0, 1, 2, 3). \tag{4}$$

For the metrics (3) the latter equation takes the form:

$$\begin{aligned} &\left(1 - \frac{r_h}{r}\right)^{-2-2\delta}\left(\frac{\partial S}{c\partial t}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 - \\ &- \left(1 - \frac{r_h}{r}\right)\left[1 - \frac{r_h}{r}\left(1 - \frac{r_h}{r}\right)^{-\delta}\right]\left(\frac{\partial S}{\partial r}\right)^2 = m^2c^2. \end{aligned} \tag{5}$$

Since the metrics does not depend explicitly on time t and the angle φ, we look for a solution of equation (5) in the form:

$$S = -Et + L\varphi + S_r(r), \tag{6}$$

where E and L are conserving energy and projection of angular momentum on z axis, respectively. Substituting form (6) to equation (5) we find $S_r(r)$ and obtain the solution:

$$\begin{aligned} S &= -Et + L\varphi + \\ &+ \int \sqrt{\frac{\left(1 - \dfrac{r_h}{r}\right)^{-2-2\delta}\dfrac{E^2}{c^2} - \dfrac{L^2}{r^2} - m^2c^2}{\left(1 - \dfrac{r_h}{r}\right)\left[1 - \dfrac{r_h}{r}\left(1 - \dfrac{r_h}{r}\right)^{-\delta}\right]}}\, dr. \end{aligned} \tag{7}$$

Substituting then the partial derivatives into the equalities

$$p^i = m\frac{dx^i}{ds} = g^{ik}p_k = -g^{ik}\frac{\partial S}{\partial x^k} \tag{8}$$

we find the equations of motion explicitly:

$$\begin{aligned}
m\frac{dt}{ds} &= \frac{E}{c^2\left(1-\frac{r_h}{r}\right)^{2+2\delta}}, \\
m\frac{dr}{ds} &= \left[\frac{E^2}{c^2\left(1-\frac{r_h}{r}\right)^{2+2\delta}} - \frac{L^2}{r^2} - m^2c^2\right]^{1/2} \times \\
&\times \left[1-\frac{r_h}{r}\left(1+\left(1-\frac{r_h}{r}\right)^{1-\delta}\right)\right]^{1/2}, \\
m\frac{d\varphi}{ds} &= \frac{L}{r^2}.
\end{aligned}$$

Note that we consider the motion in the equatorial plane so that the fourth equation (for θ) is unnecessary.

3 Perihelion precession

3.1 Trajectory based consideration

We suggest that one revolution is a motion of a body from pericenter to pericenter. If the orbit is not closed the change of angle φ, which corresponds to this motion may be both grater or less than 2π. As usual [9], the orbit equation comes from the relation: $\partial S/\partial L = const = \varphi_0$. Thus, the following formula yields the perihelion precession $\Delta\varphi$:

$$\pi + \frac{\Delta\varphi}{2} = \int_{r_{min}}^{r_{max}} \frac{Ldr}{r^2\sqrt{A(r)}}, \tag{9}$$

where

$$\begin{aligned}
A(r) = {} & \frac{1-\frac{r_h}{r}\left(1-\frac{r_h}{r}\right)^{-\delta}}{\left(1-\frac{r_h}{r}\right)^{1+2\delta}}\frac{E^2}{c^2} - \\
& - \left(\frac{L^2}{r^2}+m^2c^2\right)\left(1-\frac{r_h}{r}\right)\left[1-\frac{r_h}{r}\left(1-\frac{r_h}{r}\right)^{-\delta}\right].
\end{aligned}$$

The energy E also includes the rest mass. We make the substitution $E \to E + mc^2$ to exclude the rest mass from E and obtain:

$$\begin{aligned}
A(r) &= \frac{1-\frac{r_h}{r}\left(1-\frac{r_h}{r}\right)^{-\delta}}{\left(1-\frac{r_h}{r}\right)^{1+2\delta}}\left(\frac{E^2}{c^2}+2Em\right) - \\
&- \frac{L^2}{r^2}\left(1-\frac{r_h}{r}\right)\left[1-\frac{r_h}{r}\left(1-\frac{r_h}{r}\right)^{-\delta}\right] + \\
&+ m^2c^2\left[\frac{1}{\left(1-\frac{r_h}{r}\right)^{1+2\delta}} - 1 + \frac{r_h}{r}\right] \times \\
&\times \left[1-\frac{r_h}{r}\left(1-\frac{r_h}{r}\right)^{-\delta}\right].
\end{aligned} \tag{10}$$

Since we consider the long-distance motion, i.e. $r_h/r \ll 1$, we should retain only the terms that are not less than $(r_h/r)^2$. Taking into account the estimations:

$$\frac{E}{mc^2} \sim \frac{L^2}{m^2c^2r^2} \sim \frac{r_h}{r}, \tag{11}$$

we obtain with required accuracy:

$$\begin{aligned} A(r) &= 2m\left(E + \frac{(mc^2(1+\delta) + 2E\delta)r_h}{r}\right) - \frac{L^2}{r^2} + \\ &+ \frac{E^2}{c^2} + \frac{(1+\delta)(2\delta-1)m^2c^2r_h^2}{r^2} + \frac{2L^2r_h}{r^3}, \end{aligned}$$

or

$$A(r) = 2m\left(E + \frac{\alpha}{r}\right) - \frac{L^2}{r^2} + \frac{E^2}{c^2} - \frac{2m\beta}{r^2} - \frac{2m\gamma}{r^3}, \tag{12}$$

where

$$\begin{aligned} \alpha &= (mc^2(1+\delta) + 2E\delta)r_h, \\ \beta &= -\frac{1}{2}(1+\delta)(2\delta-1)mc^2r_h^2, \\ \gamma &= -\frac{L^2r_h}{m}. \end{aligned} \tag{13}$$

In this section we calculate $\Delta\varphi$ by expanding the integral in equation (9) with respect to the small terms β/r^2 and γ/r^3 [9]. Thus, we obtain

$$\Delta\varphi = -\frac{2\pi\beta m}{L^2} - \frac{6\pi\alpha\gamma m^2}{L^4}. \tag{14}$$

Using relations (13) we have:

$$\begin{aligned} \Delta\varphi &= \frac{(1+\delta)(2\delta+5)\pi m^2c^2r_h^2}{L^2} \times \\ &\times \left(1 + \frac{12\delta}{(1+\delta)(2\delta+5)}\frac{E}{mc^2}\right). \end{aligned}$$

The second term in the brackets is small in the framework of the considered approximation and should be omitted. Then we have

$$\Delta\varphi = \frac{(1+\delta)(2\delta+5)\pi m^2c^2r_h^2}{L^2}. \tag{15}$$

3.2 Action based consideration

To calculate the perihelion precession we can also follow the method developed in [10] and consider the action S instead of the equations of motion or the trajectory.

Let us denote $x = r_h/r$. The factor in the term with L^2/r^2 under the square root in equation (7) is

$$f_1(x) = (1-x)^{-1}\left[1 - x\left(1-x\right)^{-\delta}\right]^{-1} \tag{16}$$

Expansion to the linear order yields

$$f_1(x) = 1 + 2x + \ldots \tag{17}$$

Then the term with L^2/r^2 in the integral in equation (7) is approximately

$$-\frac{L^2}{r^2}\cdot\left(1 + 2\frac{r_h}{r}\right). \tag{18}$$

Then we introduce a new variable r':

$$\frac{1}{r'^2} = \frac{1}{r^2} \cdot \left(1 + 2\frac{r_h}{r}\right), \tag{19}$$

or

$$r = r' + r_h. \tag{20}$$

In terms of r' the term with L^2/r^2 under the integral takes the form $- L^2/r'^2$. Then we calculate how the change of variables (20) influences other terms. One should first expand each factor in equation (7) to the second order:

$$\begin{aligned} &\left(1 - \frac{r_h}{r' + r_h}\right)^{-1} \left[1 - \frac{r_h}{r' + r_h}\left(1 - \frac{r_h}{r' + r_h}\right)^{-\delta}\right]^{-1} \sim \\ &\sim 1 + 2\,\frac{r_h}{r'} + (1 + \delta) \cdot \frac{r_h^2}{r'^2} \end{aligned} \tag{21}$$

$$\begin{aligned} &\left(1 - \frac{r_h}{r' + r_h}\right)^{-3-2\delta} \left[1 - \frac{r_h}{r' + r_h}\left(1 - \frac{r_h}{r' + r_h}\right)^{-\delta}\right]^{-1} \sim \\ &\sim 1 + (4 + 2\delta) \cdot \frac{r_h}{r'} + \left(2\delta^2 + 8\delta + 6\right) \cdot \frac{r_h^2}{r'^2} \end{aligned} \tag{22}$$

As in the trajectory method (subsection 3.1) we exclude the rest mass by substitution $E \to E + mc^2$ and for the sake of brevity omit the prime in r'. Then the under-root expression takes the form:

$$\begin{aligned} &\frac{(E + mc^2)^2}{c^2}\left[1 + (4 + 2\delta) \cdot \frac{r_h}{r} + \left(2\delta^2 + 8\delta + 6\right) \cdot \frac{r_h^2}{r^2}\right] - \\ &- \frac{L^2}{r^2} - m^2c^2\left[1 + 2\,\frac{r_h}{r} + (1 + \delta) \cdot \frac{r_h^2}{r^2}\right]. \end{aligned} \tag{23}$$

After opening the brackets, ordering the degrees of r_h/r and taking into account estimations (11) and the fact that $E \ll mc^2$ and $r_h/r \ll 1$, the latter expression can be written with required accuracy as:

$$\begin{aligned} &\left(\frac{E^2}{c^2} + 2Em\right) + 2\left[2Em\,(2 + \delta) + m^2c^2\,(1 + \delta)\right]\frac{r_h}{r} + \\ &+ m^2c^2\,(1 + \delta)\,(5 + 2\delta)\,\frac{r_h^2}{r^2} - \frac{L^2}{r^2}. \end{aligned}$$

The action S itself takes the form:

$$\begin{aligned} S &= -Et + L\varphi + \int \Bigg[\left(\frac{E^2}{c^2} + 2Em\right) + \\ &+ 2\left[2Em\,(2 + \delta) + m^2c^2\,(1 + \delta)\right]\frac{r_h}{r} - \\ &- \frac{1}{r^2}\left(L^2 - m^2c^2\,(1 + \delta)\,(5 + 2\delta)\,r_h^2\right)\Bigg]^{1/2} dr. \end{aligned} \tag{24}$$

As it is well known, the correction factors in first two terms in the integrand in equation (24) cause only the correspondence between the particle energy, angular momentum and parameters

of its Kepler ellipse. The change in factor in front of $1/r^2$ leads to the systematic *secular* precession of the orbit perihelion.

As in the first method, the trajectory is determined by the equation:

$$\varphi + \frac{\partial S_r}{\partial L} = const, \tag{25}$$

where $S_r(r)$ is a radial part of the action (i.e. the integral in eq. (24)). Hence, the change of angle φ after one revolution (from perihelion to perihelion) is

$$2\pi + \Delta\varphi = -\frac{\partial}{\partial L}\Delta S_r, \tag{26}$$

where ΔS_r is the respective change of S_r. Expanding S_r with respect to the small correction factor in front of $1/r^2$, we obtain:

$$\Delta S_r = \Delta S_r^{(0)} - \frac{m^2c^2\,(1+\delta)\,(5+2\delta)\,r_h^2}{2L}\cdot\frac{\partial \Delta S_r^{(0)}}{\partial L}. \tag{27}$$

Then we differentiate the latter relation with respect to L and take into account that

$$-\frac{\partial}{\partial L}\Delta S_r^{(0)} = \Delta\varphi^{(0)} = 2\pi. \tag{28}$$

We also neglect the second derivative $\partial^2\Delta S_r^{(0)}/\partial L^2$. As a result, we find:

$$\Delta\varphi = \frac{(1+\delta)\,(5+2\delta)\,\pi m^2c^2r_h^2}{L^2}. \tag{29}$$

Certainly, this result coincides with expression (15).

4 Deflection of light

To calculate the deflection of light we also use the Hamilton-Jacobi method. Henceforth, we follow the technique set out in [10]. We first write the Hamilton-Jacobi equation for the eikonal ψ (obviously, $m = 0$ for light):

$$g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k} = 0. \tag{30}$$

As in Sect. 2 we look for a solution in the form:

$$\psi = -\omega_0 t + L\varphi + \psi_r(r), \tag{31}$$

where ω_0 is frequency of the light observed at infinity.

Thus, for ψ_r we obtain:

$$\psi_r = \frac{\omega_0}{c}\int\sqrt{\frac{\left(1-\frac{r_h}{r}\right)^{-2-2\delta} - \frac{\rho^2}{r^2}}{\left(1-\frac{r_h}{r}\right)\left[1-\frac{r_h}{r}\left(1-\frac{r_h}{r}\right)^{-\delta}\right]}}\,dr, \tag{32}$$

where the notation $\rho = cL/\omega_0$ is introduced. The expansion of the integrand to the first order in r_h/r yields

$$\psi_r = \frac{\omega_0}{c}\int\sqrt{1-\frac{\rho^2}{r^2}+2\frac{r_h}{r}\left((2+\delta)-\frac{\rho^2}{r^2}\right)}\,dr. \tag{33}$$

Expanding the integral with respect to r_h/r we further obtain:

$$\begin{aligned} \psi_r &= \psi_r^{(0)} + \frac{(2+\delta) r_h \omega_0}{c} \int \frac{dr}{\sqrt{r^2 - \rho^2}} - \\ &- \frac{r_h \rho^2 \omega_0}{c} \int \frac{dr}{r^2 \sqrt{r^2 - \rho^2}}, \end{aligned}$$

where $\psi_r^{(0)}$ corresponds to the free (straight) propagation of the light. Evaluating the integrals with the limits from ρ to a large distance R, we obtain the eikonal change $\Delta\psi_r$ explicitly:

$$\begin{aligned} \Delta\psi_r &= \Delta\psi_r^{(0)} + \frac{2(2+\delta) r_h \omega_0}{c} \operatorname{Arcosh} \frac{R}{\rho} - \\ &- \frac{2 r_h \omega_0}{c} \sqrt{1 - \frac{\rho^2}{R^2}}. \end{aligned} \tag{34}$$

From equation (31) one can see that the light deflection $\Delta\theta$ is

$$\Delta\theta = -\frac{\partial \Delta\psi_r}{\partial L}. \tag{35}$$

Using equation (34) and making R infinite after evaluation the derivative with respect to L we finally obtain:

$$\Delta\theta = \frac{2(2+\delta) r_h}{\rho}. \tag{36}$$

5 Discussion

According to the paper [5] $r_h = GM/c^2$. In the classical limit the metrics (3) yields Newtonian potential α/r, where $\alpha = (1+\delta)GMm$ (see expr. (13)). This implies that an observer would measure that the body orbits a gravitating center with the mass $M_0 = (1+\delta)M$. By the way, this potential α/r immediately yields the gravitational redshift, viz. $\Delta\nu/\nu = -GM_0/c^2 r_0$, for a photon emitted at radius r_0 and registered at infinity. However, this effect does not allow us to distinguish a wormhole from a black hole.

Finally,

$$\Delta\varphi = \frac{2\delta + 5}{1+\delta} \frac{\pi G^2 M_0^2 m^2}{c^2 L^2}, \tag{37}$$

where M_0 is a mass of a wormhole, measured by a distant observer.

The perihelion precession $\Delta\varphi_{BH}$ for a body orbiting a black hole with mass M_0 is given by the formula [10]:

$$\Delta\varphi_{BH} = \frac{6\pi G^2 M_0^2 m^2}{c^2 L^2}. \tag{38}$$

Making use of equation (37) we obtain the ratio

$$\frac{\Delta\varphi}{\Delta\varphi_{BH}} = \frac{2\delta + 5}{6(1+\delta)} = \frac{1}{3} + \frac{1}{2(1+\delta)}. \tag{39}$$

Usually δ is considered to be small and positive [5]. We assume $0 < \delta < 1$. Then, from the latter equation it is clear that $\Delta\varphi$ is always smaller than $\Delta\varphi_{BH}$. $\Delta\varphi$ ranges in the interval:

$$0.6\Delta\varphi_{BH} \lesssim \Delta\varphi \lesssim 0.8\Delta\varphi_{BH}. \tag{40}$$

It means that the smallest difference is about 20%.

Analogously, using the experimentally measured mass M_0 in the light deflection formula (36), we obtain:

$$\Delta\theta = \frac{2+\delta}{1+\delta}\frac{r_g}{\rho}, \tag{41}$$

where $r_g = 2GM_0/c^2$ (as if the gravitating center were a BH with gravitational radius r_g). The black hole light deflection amounts to [11, 10]:

$$\Delta\theta_{BH} = 2\frac{r_g}{\rho}. \tag{42}$$

Hence, we obtain for the worm to black hole ratio:

$$\frac{\Delta\theta}{\Delta\theta_{BH}} = \frac{2+\delta}{2(1+\delta)} = \frac{1}{2} + \frac{1}{2(1+\delta)}. \tag{43}$$

Therefore, $\Delta\theta$ ranges in the interval

$$0.75\Delta\theta_{BH} \le \Delta\theta \le \Delta\theta_{BH}. \tag{44}$$

The difference may amount to 25%.

The magnitude of the calculated effects is of order of several tens of percents. This gives a hope that in the not-so-distant future these differences will possibly be registered, and one will be able to answer the question whether the wormholes are encountered or not in some astrophysical objects.

The deflection of light in the gravitation field of the Sun during total eclipses was measured with at least one percent precision as well as Mercury's perihelion precession. It means that most probably there is no wormhole inside the Sun.

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References

[1] A. Einstein, and N. Rosen, Phys. Rev. **48**, 73 (1935).

[2] M. Visser, "Lorentzian Wormholes: from Einstein to Hawking", United Book, Baltimor, USA, 1996.

[3] M.S. Morris and K.S. Thorn, Amer. J. Phys. **56**, 395 (1988).

[4] F.S.N. Lobo, Phys. Rev. D **71**, 084011 (2005).

[5] N.S. Kardashev, I.D. Novikov, and A.A. Shatskiy, Astron. Rep. **50**, 601 (2006), astro-ph/0610441.

[6] A. Shatskiy, Astron. Rep. **48**, 7 (2004).

[7] J.G. Cramer, R.L. Forward, M.S. Morris, M. Visser, G. Benford, and G.A. Landis, Phys. Rev. D **51**, 3117 (1995).

[8] P.F. González-Díaz, Phys. Lett. B **632**, 159 (2006).

[9] L.D. Landau and E.M. Lifshitz, "Mechanics", Pergamon, Oxford, 1975.

[10] L.D. Landau and E.M. Lifshitz, "The Classical Theory of Fields", Pergamon, Oxford, 1975.

[11] A. Einstein, Ann. Phys. **49**, 769 (1916).